5.2 The Eigenvalue Method for Homogeneous Systems

want to solve $x_1'(t) = a x_1 + b x_2$ goal: $x_1(t) = ?$

$x_2'(t) = c x_1 + d x_2$ $x_2(t) = ?$

as a matrix equation: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

form: $\vec{x}\,' = A\vec{x}$ a, b, c, d constants

solutions: $\vec{x} = \vec{v} e^{\lambda t}$ ($n \times n$ $A \to n$ of these)

$\lambda \to$ eigenvalue of A

$\vec{v} \to$ corresponding eigenvector

general solution: linear combination of all

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \cdots$$

Solve $A\vec{v} = \lambda\vec{v}$ for $\lambda, \vec{v}$

Solve $\boxed{\det(A - \lambda I) = 0 \text{ for } \lambda}$

then solve $\boxed{(A - \lambda I)\vec{v} = \vec{0}}$ using the λ's from above $\vec{v}$

<u>example</u>

$$x_1' = x_1 + 2x_2$$

$$x_2' = 3x_1 + 2x_2$$

$$\vec{x}\,' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \vec{x}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0 \qquad \text{characteristic eq.}$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = -1, \quad \lambda = 4$$

now the corresponding eigenvectors

Solve $(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v}$

$\lambda = -1$: $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \vec{v} = \vec{0} \qquad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} 2 & 2 & 0 \\ 3 & 3 & 0 \end{bmatrix} \quad \text{row reduction}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 3 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b is free, $a + b = 0$

$\vec{v} = \begin{bmatrix} -b \\ b \end{bmatrix}$ choose any $b \neq 0$, e.g. $b = -1$

$$\boxed{\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda = -1}$$

similarly, the other pair is $\boxed{\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \lambda = 4}$

$$\boxed{\text{general solution: } \vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$\vec{v}e^{\lambda t}$

c_1, c_2 come from initial conditions

for example, $x_1(0) = 1$, $x_2(0) = 0$

then at $t = 0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

solve ... $c_1 = -\frac{3}{5}$ $\quad c_2 = \frac{1}{5}$

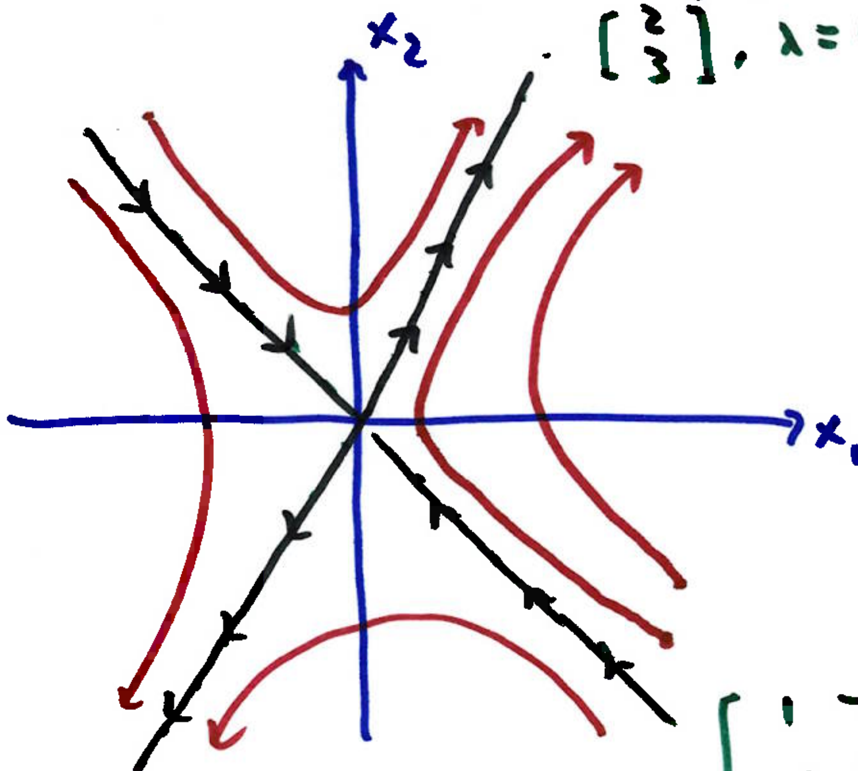
graph x_1 vs x_2 → Phase Diagrams or Phase Portraits

$$\vec{x} = c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{4t}\begin{bmatrix}2\\3\end{bmatrix}$$

as $t \to \infty$, $\vec{x} \to c_2 e^{4t}\begin{bmatrix}2\\3\end{bmatrix}$ → parallel to vector $\begin{bmatrix}2\\3\end{bmatrix}$

as $t \to -\infty$, $\vec{x} \to c_1 e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}$ → " " " $\begin{bmatrix}1\\-1\end{bmatrix}$

eigenvectors act like asymptotes to solution curves



$\begin{bmatrix}2\\3\end{bmatrix}$, $\lambda = 4$ → $\lambda > 0$ moving <u>away</u> origin

$\begin{bmatrix}1\\-1\end{bmatrix}$, $\lambda = -1$ → $\lambda < 0$ moving <u>towards</u> origin example

$x_1' = -3x_1 + x_2$

$x_2' = x_1 - 3x_2$

$$\vec{x}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{x}$$

suppose we found $\lambda = -4$, $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = -2$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

general solution: $\vec{x} = c_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

asymptotes: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ solutions go toward origin $(\lambda < 0)$

when $t > 0$, $e^{-2t} > e^{-4t}$

so, solutions follow eigenvector of $\lambda = -2 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$t < 0$, opposite situation $\rightarrow$ follow $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

origin is t at ∞ since both λ's are negative

solutions start w/ $t<0$ then increases to $t=\infty$

$\rightarrow$ follow $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$